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Symmetry Breaking in Hamiltonian Systems

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INTRODUCTION

Let: $(p, q) \in \mathbb{R}^{2N}$, $H: \mathbb{R}^{2N} \to \mathbb{R}$ and h_1, h_2 be T-periodic functions ($T > 0$) from $\mathbb{R}$ to $\mathbb{R}^N$. In this paper we look for T-periodic solutions of the Hamiltonian system

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} + \varepsilon h_1(t) \\ \dot{q} &= \frac{\partial H}{\partial p} + \varepsilon h_2(t) \end{aligned} \tag{H_ε}$$

for $\varepsilon > 0$, small enough.

Here the unperturbed system (H_0) is autonomous and hence S^1-invariant, while this is no more true for (H_ε), $\varepsilon > 0$, in view of the forcing terms h_1 and h_2.

From the abstract point of view this problem leads to the investigation of the existence of critical points for perturbations f_ε of a functional f whose critical points appear in manifolds.

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The paper is divided into five sections.

Sections 1 and 2 deal with the abstract setting, namely, perturbation in critical point theory. In the former we suppose that the unperturbed functional f has a minimum consisting of a manifold of critical points Z. Assuming that Z has two non-trivial homology groups, we show (Theorem 1.2) that f_ε has at least two critical points at levels near $f(Z)$. Such a result is obtained using methods of Morse theory, but does not require any nondegeneracy assumption on f.

In Section 2 we consider the general case when Z is a compact, connected manifold of critical points of f, possibly not at the minimum level. The method consists of showing that the critical points of f can be found as critical points of the restriction $f_\varepsilon|_{Z_\varepsilon}$, where Z_ε is a compact manifold diffeomorphic to Z. After completing the work we found out that such a method has already been used in [15]; see also [3, 6, 17, 18].

In Section 3 we study (H_ε). The use of the Dual Action Principle [4, 5] allows us to use the above abstract results in order to find forced oscillations of (H_ε). Theorem 3.6 shows that near a nondegenerate orbit of (H_0) there are at least two forced oscillations of (H_ε). The nondegeneracy assumption is lifted in Theorem 3.4. In Theorem 3.7 we show that if (H_0) is completely integrable, the number of forced oscillations increases to $N+1$.

In Section 4 we locate the bifurcating orbits of (H_ε) near the corresponding orbit of (H_0). Finally, in Section 5, we investigate the (linear) stability of the forced oscillations. We show, for instance, that if an orbit of (H_0) is nondegenerate and stable, then one of the bifurcating orbits is stable and the other one unstable, as usual in bifurcation theory.

Our results in Section 1 are related to those of [12], which investigates the situation where f has nondegenerate critical points (in the usual sense) and f_ε is a C^0 perturbation of f. Here, however, the critical points we investigate are degenerate (they come in manifolds) and we are looking for multiplicity results.

Finally, we point out that our results on (H_ε) improve those of [1]. In particular, the same "necessary conditions" of Section 4 were already obtained there.

1. Perturbation of Minima

Let E be a Hilbert space with norm $\|\cdot\|$ and let $f \in C^1(E, \mathbb{R})$. We set $f^b = \{x \in E: f(x) \leqslant b\}$, $K = K(f) = \{x \in E: f'(x) = 0\}$, where f' denotes grad f, and $K_b = K_b(f) = \{x \in K: f(x) = b\}$. A critical level for f is a $c \in \mathbb{R}$ such that $K_c \neq \varnothing$.

On f we will assume

$$c := \inf_E f > -\infty \tag{1.1}$$

and the so-called condition (C) of Palais and Smale

every sequence $\{x_j\} \subset E$ such that $f(x_j)$ is bounded and $f'(x_j) \to 0$ has a converging subsequence. (PS)

It is well known that under such assumptions c is a critical level for f and $c = \min_E f$; see [14].

Further, we suppose:

$$\text{There is } q^* > 0 \text{ such that } H_{q^*}(K_c(f)) \neq \{0\}. \tag{1.2}$$

Here and in the sequel H_q denotes the qth homology group with coefficients in a group $\mathscr{G}$.

The example below shows that (1.2) is actually a symmetry condition.

EXAMPLE 1.1. Suppose S^1 acts on E through an action A_θ under which f is invariant, namely: $f(A_\theta u) = f(u)$ $\forall u \in E$, $\forall \theta \in S^1$. For all $u \in K_c$, let $\Omega_u = \{v \in E\colon v = A_\theta u \text{ for some } \theta \in S^1\}$. Clearly $\Omega_u \subset K_c$. If, further, $K_c \cap \mathrm{Fix}(S^1) = \varnothing$ and there exist $u_1, \ldots, u_k \in K_c$ such that

$$K_c = \Omega_{u_1} \cup \cdots \cup \Omega_{u_k} \qquad \text{with } \Omega_{u_i} \cap \Omega_{u_j} = \varnothing \text{ for } i \neq j,$$

then each $\Omega_{u_i} \simeq S^1$ and hence

$$H_q(K_c) \simeq \bigoplus_{i=1}^{k} H_q(\Omega_{u_i}) \simeq \bigoplus_{i=1}^{k} H_q(S^1).$$

Thus (1.2) holds, taking $q^* = 1$. ∎

We are in position to state:

THEOREM 1.2. *Let $f, g \in C^1(E, \mathbb{R})$ satisfy* (PS) *and* (1.1), *with $c = \min_E f$, $\gamma = \min_E g$. Further, we suppose* (1.2) *holds for f and that there exists $\delta > 0$ such that*

$$K_b(f) = \varnothing \qquad \forall c < b \leqslant c + \delta; \tag{1.3}$$

$$f^c \subset g^{c+\delta/2} \subset f^{c+\delta}. \tag{1.4}$$

Then g has at least two critical points in $g^{c+\delta/2}$.

Proof. First we recall [12] that $f^c = K_c(f)$ is a deformation retract of $f^{c+\delta}$ because of (1.3) and (PS). Remark that such a deformation can still be constructed in the present regularity assumptions using the "Pseudo-gradient vector field" [14]. Hence:

$$H_q(f^{c+\delta}) \simeq H_q(K_c(f)) \qquad \forall q \geqslant 0. \tag{1.5}$$

Next, let $u^* \in K_\gamma(g)$ $(\neq \varnothing)$. Remark that $\gamma \leqslant c + \frac{1}{2}\delta$. Suppose, by contradiction, that $K(g) \cap g^{c+\delta/2} = \{u^*\}$. Then, by the same arguments recalled before, g^γ is a deformation retract of $g^{c+\delta/2}$ and

$$H_q(g^{c+\delta/2}) \simeq H_q(g^\gamma) \simeq H_q(\{u^*\}) \qquad \forall q \geqslant 0. \tag{1.6}$$

In particular, $H_q(g^{c+\delta/2}) = \{0\}$ $\forall q > 0$. From (1.4) we deduce the commutativity of the diagram

$$\begin{tikzcd} H_q(f^c) \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & H_q(f^{c+\delta}) \\ H_q(g^{c+\delta/2}) \arrow[ur, "\phi_3"'] & \end{tikzcd}$$

ϕ_i being the canonical homomorphisms induced by the inclusions $f^c \subset g^{c+\delta/2} \subset f^{c+\delta}$.

Since ϕ_1 is an isomorphism (see (1.5)), we deduce that $\forall q$ such that $H_q(g^{c+\delta/2}) = \{0\}$ it follows that $H_q(f^c) = \{0\}$. Hence, by (1.6), one has

$$H_q(f^c) = \{0\} \qquad \forall q > 0,$$

in contradiction with (1.2). ∎

Remark 1.3. Let us point out that, since $c = \min f$, no assumptions of nondegeneracy are made on f (same for g). Moreover, we remark that g is a C^0 perturbation of f, according to (1.4).

With respect to Example 1.1, g does not satisfy, possibly, any "symmetry condition"; this will be the case in the applications of Section 3.

The above arguments can be carried out–even if in a less general setting—for any critical level for f. More precisely, let $f, g \in C^2(E, \mathbb{R})$ satisfy (PS) and let b be an isolated level for f. Suppose:

$$\text{there are } q_1 \neq q_2 \text{ such that } H_{q_i}(f^b, f^b - K_b) \neq \{0\}, \qquad i = 1, 2, \tag{1.7}$$

and that $\exists \delta > 0$ such that

$$f^{b-\delta} \subset g^{b-\delta/2} \subset f^{b-\delta/6} \subset f^{b+\delta/6} \subset g^{b+\delta/2} \subset f^{b+\delta}. \tag{1.8}$$

Then it is possible to show that g has at least a critical point in $X = \{b - \frac{1}{2}\delta \leqslant g \leqslant b + \frac{1}{2}\delta\}$. Moreover, if g is a *Morse functional*, then one can show that g actually has at least two critical points in X, one for each q_i, $i = 1, 2$, given in (1.7).

Besides the regularity assumptions, the main disadvantage of the above result is that we need to suppose the perturbation g is a Morse functional.

In the following we will show, by different arguments, that the results are true in general.

2. Perturbation near a Nondegenerate Critical Manifold

Here we will deal with a family $f_\varepsilon(u) = f(\varepsilon, u)$ of functionals. For the sake of simplicity in the exposition and to avoid technicalities, we will assume f_ε is defined on a *Hilbert space* E and is smooth with respect to $(\varepsilon, u) \in \mathbb{R} \times E$. We will write f for $f(0, \cdot)$, $f'(\varepsilon, u)$ for the gradient $\nabla_u f(\varepsilon, u)$, $f'(u)$ for $f'(0, u)$, $f''(u)$ for the second derivative $D_{uu} f(0, u)$, etc.

Let Z be such that:

$$Z \subset K(f) \text{ is a compact, connected manifold;} \tag{2.1}$$

$$\forall z \in Z, \qquad T_z Z = \operatorname{Ker}(f''(z)). \tag{2.2}$$

Here and in the following by "manifold" we mean a smooth manifold without boundary and $T_x M$ denotes the tangent space to the manifold M at $x \in M$.

We will refer to a manifold Z satisfying (2.1)–(2.2) as a *compact, nondegenerate critical manifold.*

Actually, (2.2) is a nondegeneracy condition (see Section 3 below) which turns out to be slightly weaker than the one in [9].

Let $\operatorname{cat}(Z)$ denote the Lusternik–Schnirelman (LS) category of Z with respect to itself, namely the least integer k such that $Z \subset X_1 \cup \cdots \cup X_k$, each X_i closed subset of Z, contractible to a point in Z. Recall that the LS category is a topological invariant. Moreover, every $\phi \in C^2(M, \mathbb{R})$, M smooth compact manifold, has at least $\operatorname{cat}(M)$ critical points. In fact a much more general result holds true; see [14].

The following results holds:

Theorem 2.1. *Let Z be a compact nondegenerate critical manifold for f and suppose $f''(z)$ is a Fredholm operator of index* 0 *for all $z \in Z$. Then there exist $\bar{\varepsilon} > 0$ and a neighborhood U of Z such that for all $0 < |\varepsilon| < \bar{\varepsilon}$, f_ε has at least* $\operatorname{cat}(Z)$ *critical points in U.*

The above result is essentially contained in [15]; see also [3, 6, 18]. We give below a sketch of the proof, for the reader's convenience only. For $z \in Z$ we set $N_z = \operatorname{Ker}\{f''(z)\}$ and $R_z = \operatorname{Range}\{f''(z)\}$. Since $f''(z)$ is a Fredholm map of index 0, then $\forall z \in Z$ we can write $E = N_z \oplus R_z$. First of all, an application of the local inversion theorem leads to:

LEMMA 2.2. *There exist a neighborhood U of Z and smooth mappings $P: U \to E$ and $Q: U \to E$, such that $\forall u \in U$,*

$$P(u) \in Z, \tag{2.3}$$

$$Q(u) \in R_{Pu}, \tag{2.4}$$

$$u = Pu + Qu. \tag{2.5}$$

Next, one has:

LEMMA 2.3. *Let U be as in the preceding lemma. Then here is an $\bar{\varepsilon} > 0$ such that for all $0 < |\varepsilon| = \bar{\varepsilon}$ the set*

$$Z_\varepsilon = \{u \in U: f'(\varepsilon, u) \in \operatorname{Ker}\{f''(Pu)\}\}$$

is diffeomorphic to Z. In particular:

(i) *Z_ε is a smooth, compact, connected manifold;*

(ii) $\operatorname{cat}(Z_\varepsilon) = \operatorname{cat}(Z)$.

Remark 2.4. Since $f(\varepsilon, u)$ depends smoothly on ε, then $T_u Z_\varepsilon$ is close to $T_{Pu} Z$ for ε small.

We now show:

LEMMA 2.5. *For ε small $f'(\varepsilon, u) = 0$ iff $u \in Z_\varepsilon$ and u is a critical point of f_ε constrained on Z_ε.*

Proof. Let $u \in Z_\varepsilon$ be a critical point for the restriction $f_\varepsilon|_{Z_\varepsilon}$. This means that $f'(\varepsilon, u) \in (T_u Z_\varepsilon)^\perp$. On the other hand, in view of the definition of Z_ε and (2.2), one has $f'(\varepsilon, u) \in (T_{Pu} Z)$. Since $T_u Z_\varepsilon$ and $T_{Pu} Z$ are close (see Remark 2.4), taking possibly ε smaller, it follows that $f'(\varepsilon, u) = 0$. The converse is trivial. ∎

Proof of Theorem 2.1. By Lemma 2.5 it is enough to look for the critical points of f_ε constrained on Z_ε.

According to the LS critical point theory (see the remark before the statement of Theorem 2.1) we use Lemma 2.3 to deduce that f_ε has on Z_ε at least $\operatorname{cat}(Z_\varepsilon) = \operatorname{cat}(Z)$ critical points. ∎

Remark 2.6. In the applications in Section 3 we shall work in a Banach space E. It is easy to modify the above arguments to cover such a case.

Remark 2.7. In the case in which f_ε breaks only some of the symmetries of f, namely if f_ε and Z_ε are still invariant under some group action, the statement of Theorem 2.1 can be sharpened. Precisely, if $\sim$ denotes the equivalence induced by such an action, f_ε will possess at least $\operatorname{cat}(Z_\varepsilon/\sim)$ critical points near Z (provided $Z_\varepsilon/\sim$ makes sense as a manifold).

Remark 2.8. We points out that the fact that f_ε has at least cat(Z) critical points has been observed in various, but different settings, e.g., [8, 17, 18].

3. Forced Oscillations of Hamiltonian Systems

We now apply the abstract results of the preceding sections to time-dependent perturbations of convex, autonomous Hamiltonian systems. We investigate systems of the following kind:

$$-J\dot{z} = H'(z) + \varepsilon h(t), \tag{H_ε}$$

where $z = (p, q) \in \mathbb{R}^{2N}$, H' is the gradient of H and J is the symplectic matrix defined by $J(p, q) = (-q, p)$.

We assume that h is T-periodic, and we look for T-periodic solutions of (H_ε) when ε is small.

We first apply the results of Section 1. For this, we need the following assumptions:

(A1) H is C^1, strictly convex, $H(x) \geqslant H(0) = 0$, and there are constants $\gamma > 0$ and a_0 such that

$$\langle H'(x), x \rangle \geqslant \gamma |x| - a_0;$$

(A2) there are constants k and a_1 such that

$$H(x) \leqslant \frac{k}{2} |x|^2 + a_1;$$

(A3) there is a constant $K > k$ such that

$$\liminf_{x \to 0} [H(x) |x|^{-2}] \geqslant \frac{K}{2};$$

(A4) set $\bar{h} = (1/T) \int_0^T h(t)\, dt$; then there is a function $g \in L^\infty$ such that $dg/dt = h - \bar{h}$ in the sense of distributions.

Note that (A4) allows, for instance, h to be a Dirac mass, so that our analysis cover the case of the equation:

$$-J\dot{z} = H'(z) + \varepsilon \delta_{T\mathbb{Z}}(t) \xi, \qquad \xi \in \mathbb{R}^{2N} \text{ prescribed.}$$

Let G be the Fenchel conjugate [10] of H, i.e., $G(y) = \sup_{x \in \mathbb{R}^{2N}} \{\langle x, y \rangle - H(x)\}$. It follows from (A1) and (A2) that $G: \mathbb{R}^{2N} \to \mathbb{R}$

is a well defined convex function, which is strictly convex and C^1, and that $G(y) \geqslant G(0) = 0$.

Define the Hilbert space E by

$$E = \left\{ u \in L^2(0, T; \mathbb{R}^{2N}) \mid \int_0^T u \, dt = 0 \right\},$$

with norm $\|u\|^2 = \int_0^T |u|^2 \, dt$.

Define a linear, selfadjoint operator $\mathscr{L} \in L(E)$ by

$$\mathscr{L}u = v \qquad \text{iff} \quad -J\frac{dv}{dt} = u.$$

We finally define a functional f_ε on E by

$$f_\varepsilon(u) = \int_0^T [G(u - \varepsilon\bar{h}) - \varepsilon\langle u, Jg(t)\rangle - \tfrac{1}{2}\langle u, \mathscr{L}u\rangle] \, dt. \tag{3.1}$$

This functional is the sum of a convex term, which we denote by $\phi_\varepsilon(u)$, and a quadratic (hence C^∞) term $\frac{1}{2}\int \langle u, \mathscr{L}u\rangle$. We shall write $0 \in \partial f_\varepsilon(u)$ to mean $0 \in \partial\phi_\varepsilon(u) + \mathscr{L}u$.

The Dual Action Principle states that, if $0 \in \partial f_\varepsilon(u)$, then there is some $\xi_\varepsilon \in \mathbb{R}^{2N}$ such that $z_\varepsilon = u + \varepsilon Jg + \xi_\varepsilon$ is a T-periodic solution of (H_ε). Our formulation here is somewhat different from Clarke's original formulation [4], and closer to [7]. So we give a short proof.

$\partial f_\varepsilon(u) \ni 0$ translates into the pointwise equation

$$G'(u - \varepsilon\bar{h}) - \varepsilon Jg(t) - \mathscr{L}u = \xi_\varepsilon \qquad \text{a.e.} \tag{3.2}$$

where $\xi_\varepsilon \in \mathbb{R}^{2N}$ is some constant vector. We can invert G' by the Legendre reciprocity formula

$$u - \varepsilon\bar{h} = H'(\mathscr{L}u + \varepsilon Jg(t) + \xi_\varepsilon).$$

Now set $z = \mathscr{L}u + \varepsilon Jg(t) + \xi_\varepsilon$. Differentiating, we get

$$-J\dot{z} = u + \varepsilon(h - \bar{h}), \tag{3.3}$$

so that Eq. (3.2) becomes what we want:

$$-J\dot{z} = H'(z) + \varepsilon h.$$

From [5] we draw the following information:

PROPOSITION 3.1. *Assume* $2\pi K^{-1} < T < 2\pi k^{-1}$. Then $f_0 \equiv f$ *has a global minimum* $\hat{u} \neq 0$, with

$$\hat{c} := f(\hat{u}) = \min f < f(0) = 0.$$

We now prove an a priori estimate:

LEMMA 3.2. *Assume* $2\pi K^{-1} < T < 2\pi k^{-1}$. *For all* $\eta > 0$ *and* $b \in \mathbb{R}$, *there is some* $r > 0$ *such that*

$$|\varepsilon| < \eta \qquad \textit{implies} \quad f_\varepsilon^b \subset B_r := \{u \in E \mid \|u\| \leqslant r\}.$$

Proof. From Wirtinger's inequality [11, p. 184]: $\int_0^T |u|^2\, dt \leqslant (T/2\pi) \int_0^T |\dot{u}|^2\, dt$, we deduce

$$\int_0^T \langle u, \mathscr{L}u\rangle\, dt \leqslant \|u\|\, \|\mathscr{L}u\| \leqslant \frac{T}{2\pi} \|u\|^2.$$

Moreover, from (A2) it follows that

$$G(y) \geqslant \frac{1}{2k} |y|^2 - a_1 \qquad \forall y \in \mathbb{R}^{2N}.$$

Hence, if $f_\varepsilon(u) \leqslant b$, we get

$$\begin{aligned} b &\geqslant \frac{1}{2k} \|u - \varepsilon\bar{h}\|^2 - \varepsilon \|u\|\, \|g\| - \frac{1}{2}\frac{T}{2\pi} \|u\|^2 - a_1 T \\ &= \frac{1}{2}\left(\frac{1}{k} - \frac{T}{2\pi}\right) \|u\|^2 - \varepsilon \|u\|\, \|g\| + \left(\frac{\varepsilon^2}{2k} |\bar{h}|^2 - a_1\right) T. \end{aligned}$$

The coefficient of the first term is positive, and the results follow. ∎

This gives us immediately another estimate:

LEMMA 3.3. *Assume* $2\pi K^{-1} < T < 2\pi k^{-1}$. *Then there are constants* $\bar{\varepsilon} > 0$ *and* $R > 0$ *such that, whenever* $|\varepsilon| < \bar{\varepsilon}$, *we have*

$$\partial f_\varepsilon(u) \ni 0 \textit{ and } f_\varepsilon(u) < 0 \qquad \textit{implies } |z_\varepsilon(t)| \leqslant R \qquad \forall t,$$

where $z_\varepsilon = \mathscr{L}u + \varepsilon Jg + \xi_\varepsilon$ *is the corresponding solution of* (H_ε).

Proof. We have $-J\dot{z}_\varepsilon = H'(z_\varepsilon) + \varepsilon h$. Multiplying both sides by z_ε and integrating, we get

$$\int_0^T \langle H'(z_\varepsilon), z_\varepsilon\rangle\, dt = \int_0^T [\langle -J\dot{z}_\varepsilon, z_\varepsilon\rangle - \langle \varepsilon h, z_\varepsilon\rangle]\, dt.$$

By assumption (A1), this becomes

$$\int_0^T (\gamma\,|z_\varepsilon| - a_0)\,dt \leqslant \int_0^T [\langle -J\dot{z}_\varepsilon, z_\varepsilon\rangle - \langle \varepsilon h, z_\varepsilon\rangle]\,dt$$

and hence, by (3.3),

$$\begin{aligned}\int_0^T \gamma\,|z_\varepsilon|\,dT &\leqslant a_0 T + \int_0^T \langle u - \varepsilon\bar{h}, z_\varepsilon\rangle\,dt\\ &= a_0 T + \int_0^T \langle u - \varepsilon\bar{h}, \mathscr{L}u + \varepsilon Jg + \xi_\varepsilon\rangle\,dt\\ &= a_0 T + \int_0^T \langle u, \mathscr{L}u\rangle\,dt + \int_0^T \langle u, \varepsilon Jg\rangle\,dt - \varepsilon T\langle \bar{h}, \xi_\varepsilon\rangle\\ &\leqslant a_0 T + \frac{T}{2\pi}\|u\|^2 + |\varepsilon|\,\|u\|\,\|g\| + |\varepsilon|\,T|\bar{h}|\,|\xi_\varepsilon|.\end{aligned}$$

We now use Lemma 3.2 with $b=0$. This yields

$$\gamma\int_0^T |z_\varepsilon|\,dt \leqslant a_0 T + \frac{T}{2\pi} r^2 + |\varepsilon|\,r\,\|g\| + |\varepsilon|\,T|\bar{h}|\,|\xi_\varepsilon|.$$

Again substituting $z_\varepsilon = \mathscr{L}u + \varepsilon Jg + \xi_\varepsilon$ we readily obtain

$$\begin{aligned}T(\gamma - |\varepsilon|\,|\bar{h}|)\,|\xi_\varepsilon| &\leqslant T\left(a_0 + \frac{r^2}{2\pi}\right) + |\varepsilon|\,r\,\|g\| + \gamma\int_0^T [|\mathscr{L}u| + |\varepsilon|\,|Jg|]\,dt\\ &\leqslant T\left\{a_0 + \frac{r^2}{2\pi} + |\varepsilon|\,r\,\|g\|^2_{L^\infty} + |\varepsilon|\,\gamma\,\|g\|_{L^\infty} + \gamma\frac{T^{1/2}}{2\pi}r\right\}.\end{aligned}$$

It follows that, if $|\bar{\varepsilon}| < \gamma|\bar{h}|^{-1}$, $|\xi_\varepsilon|$ is uniformly bounded for $|\varepsilon| < |\bar{\varepsilon}|$. The results follow immediately. ∎

We now state our results. Let us define the S^1-action A_θ by setting $Au(\cdot) = u(\cdot + \theta)$, $\theta \in S^1$.

THEOREM 3.4. *Assume* (A1) *to* (A4) *and let* $2\pi K^{-1} < T < 2\pi k^{-1}$. *Moreover suppose* $\hat{c}$ *(given in Proposition* 3.1*) is an isolated critical level for* f *and there exists* $\hat{u} \in K_{\hat{c}}(f)$ *such that the orbit* $\Omega = \{\hat{u}(\cdot + \theta), \theta \in S^1\}$ *is isolated in* $K_{\hat{c}}(f)$. *Then there exist* δ *and* $\bar{\varepsilon} > 0$ *such that, whenever* $|\varepsilon| < \bar{\varepsilon}$ *and* $\varepsilon \neq 0$, *the system* (H_ε) *has at least two T-periodic solutions, whose corresponding critical points lie in* $f^{\hat{c}+\delta}$.

Proof. We begin by modifying out Hamiltonian H. We replace it by a Hamiltonian $\tilde{H}$ such that:

(a) H and $\tilde{H}$ coincide in the ball $\|z\| \leqslant 2R$, where R is the constant of Lemma 3.3;

(b) $\tilde{H}$ satisfies (A1) to (A3), with the same constants, and an additional property, namely, that there are constants a_2 and a_3 such that

$$|\tilde{H}'(x)| \geqslant a_2 |x| - a_3. \tag{3.4}$$

We now define $\tilde{G}$ and $\tilde{f}_\varepsilon$ accordingly. Because of property (3.4), the functional $\tilde{f}_\varepsilon$ now turns out to be C^1 on E. In addition, it has the property that for every $r > 0$ there are some $\tilde{\varepsilon}$ such that

$$\|u\| \leqslant r, \ |\varepsilon| < \tilde{\varepsilon} \qquad \text{implies } |\tilde{f}(u) - \tilde{f}_\varepsilon(u)| \leqslant \delta/3. \tag{3.5}$$

Choose r by setting $\eta = 1$ and $b = \hat{c} + \delta/2$ in Lemma 3.2. For all $\varepsilon \leqslant 1$, we have

$$\tilde{f}^{\hat{c} + \delta/2} \subset B_r. \tag{3.6}$$

Taking $\varepsilon \leqslant \min(\tilde{\varepsilon}, \bar{\varepsilon}, 1)$, let $u \in \tilde{f}^{\hat{c} + \delta/2}$. From (3.5) and (3.6) it follows that

$$\tilde{f}^{\hat{c}} \subset \tilde{f}^{\hat{c} + \delta/2} \subset \tilde{f}^{\hat{c} + \delta}.$$

In order to apply Theorem 1.2 it remains to show that (1.2) holds true. In fact, by assumption, it follows that there is a compact Y (recall that $K_{\hat{c}}$ is compact because (PS) holds) such that $K_{\hat{c}}(f) = \Omega \cup Y$, with $\Omega \cap Y = \varnothing$. Using the Mayer–Vietoris sequence (see [16]) one has

$$H_q(K_{\hat{c}}) \simeq H_q(\Omega) \oplus H_q(Y).$$

Since $\hat{c} < 0$, then $\Omega \simeq S^1$ (see Example 1.1) and

$$H_1(K_{\hat{c}}) \simeq \mathscr{G} \oplus H_1(Y).$$

It now follows from Theorem 1.2 that $\tilde{f}_\varepsilon$ has at least two critical points in $\tilde{f}_\varepsilon^{\hat{c} + \delta}$. We can always pick δ so small that $\hat{c} + \delta < 0$. The two critical points, u_1 and u_2 say, then belong to $\tilde{f}_\varepsilon^0$, and the corresponding solutions z_ε^1 and z_ε^2 of $(\tilde{H}_\varepsilon)$ both satisfy $|z_\varepsilon^i(t)| \leqslant R$ by Lemma 3.3 But then, since H and $\tilde{H}$ coincide in a neighborhood of the trajectories z_ε^i, we get

$$\dot{z}_\varepsilon^i(t) = J\tilde{H}'(z_\varepsilon^i(t)) + \varepsilon Jh(t) = JH'(z_\varepsilon^i(t)) + \varepsilon Jh(t). \quad \blacksquare$$

Remark 3.5. More in general, we can assume in Theorem 3.4 that $\hat{c}$ is isolated and there exists $q^* > 0$ such that (1.2) holds. This will be the case in Example 3.8.

As for the application of Theorem 2.1, we consider again the system (H_ε). On H we will assume:

(A5) $H \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ and $\exists c > 0$: $\langle H''(x)\, y, y \rangle \geqslant c\,|y|^2\ \forall x, y \in \mathbb{R}^{2N}$.

In fact, the results of Section 2 being local, we could assume (A5) only in a neighborhood of a given orbit of solutions of (H_0).

In the following the setting will be similar to that of [8], which we will assume familiar to the reader.

We let $E = \{u \in C^r(0, T; \mathbb{R}^{2N}): \int_0^T u\, dt = 0\}$ and define G and $\mathscr{L}$ as before.

It will be more suitable for our purpose to assume that h is of class C^r, and T-periodic and to define the dual functional by

$$f(\varepsilon, u) = \int_0^T \left[G(u - \varepsilon h) - \frac{1}{2} \langle u, \mathscr{L}u \rangle \right] dt.$$

One reverts from this formulation to the preceding one by the change of variables $u \to u + \varepsilon h - \varepsilon \bar{h}$.

Remark that the Dual Action Principle still holds. Moreover, $f_\varepsilon''(u)$ is Fredholm of index 0 because $G_\varepsilon''(u)$ is an isomorphism and $\mathscr{L}$ is compact.

Suppose $z_0(t)$ is a (non-zero) isolated solution of (H_0) and let $u_0 \in E$ be the corresponding critical point of f. Obviously Z will be the orbit $\{\{u = u_0(t + \theta), \theta \in S^1\}$ and (2.1) holds.

As for (2.2) we recall that (see Proposition 7 of [8]) $w \in \operatorname{Ker}(f''(u))(u \in Z)$, if and only if $\exists \xi \in \mathbb{R}^{2N}$ such that $v = \mathscr{L}w + \xi$ is a T-periodic solution of the linear system

$$-J\dot{v} = H''(z(t))v, \qquad z = \mathscr{L}u + \zeta \text{ for some } \zeta \in \mathbb{R}^{2N}. \tag{3.7}$$

If $z(t) = z_0(t + \theta)$, then $v(t) = \dot{z}_0(t + \theta)$ is a solution of (3.7) and assumption (2.2) requires that (3.7) have no solution linearly independent from $z_0(t + \theta)$. Hence in this case, (2.2) holds, provided that z_0 is nondegenerate in the sense of [9], namely if the nullity of z_0 is 1.

By a straightforward application of Theorem 2.1 we get:

THEOREM 3.6. *Suppose* (A5) *holds and let* z_0 *be a nondegenerate* T*-periodic solution of* (H_0). *Then there exists* $\bar{\varepsilon} > 0$ *such that for* $0 < |\varepsilon| < \bar{\varepsilon}$ *the forced system* (H_ε) *has at least two* T*-periodic solutions near* $\{z_0(t + \theta): \theta \in S^1\}$.

As a further application, we consider the case in which the Hamiltonian system (H_0) is *completely integrable.* We recall (see, for Example, [13]) that this means that there exists a canonical change of coordinates $(p, q) \to (I, \phi)$, $\phi_i \in S^1$, $i = 1, \ldots, N$, such that (H_0) is transformed into

$$\begin{aligned} \dot{I} &= 0 \\ \dot{\phi} &= K'(I) \end{aligned} \tag{3.8}$$

with $K = K(I)$ independent from ϕ. Setting $\omega(I) = K'(I)$, we consider the N-dimensional torus of solutions of (H_0) given by

$$\Sigma^N = \begin{cases} p = p(I, \omega(I) + \phi) \\ q = q(I, \omega(I) + \phi) \\ \phi = (\phi_1, \ldots, \phi_N) \in S^1 \times \cdots \times S^1 \end{cases}$$

and suppose Σ^N consists of T-periodic solutions of (H_0), which are nondegenerate, in the sense that $K''(I)$ is invertible. We point out that a linear system does not satisfy such a nondegeneracy condition.

Let Z be the (compact) critical manifold of f corresponding to Σ^N. It is easy to see, taking into account (3.8), that the above nondegeneracy of Σ^N implies that Z satisfies (2.2). Moreover, it results that $\text{cat}(Z) = N + 1$ $(= \text{cat}(\Sigma^n))$.

At this point, an application of Theorem 2.1 yields:

THEOREM 3.7. *Let (H_0) be a completely integrable system satisfying the preceding assumptions. Moreover let* (A5) *hold. Then there is $\bar{\varepsilon} > 0$ such that $\forall 0 < |\varepsilon| < \bar{\varepsilon}$ (H_ε) has at least $N + 1$ T-periodic solutions near Σ^N.*

In the following example we will show that Theorem 1.2 or Theorem 2.1 are still applicable in some cases of degeneracy (in the sense of [9]).

EXAMPLE 3.8. We take $H(z) = \phi(\frac{1}{2}|z|^2)$, with $\phi\colon \mathbb{R}^+ \mathbb{R}^+$ taken in such a way that (A1)–(A4) hold. Let z be a 2π-periodic solution of (H_0). Then it results:

$$-J\dot{z} = \phi'(\tfrac{1}{2}|z|^2)z, \qquad \phi(\tfrac{1}{2}|z|^2) = \text{constant},$$

i.e. $\frac{1}{2}|z|^2 = \rho$. Since z is 2π-periodic, then $\phi'(\rho) = k$, $k = 1, 2, \ldots$. Thus the solutions of (H_0) are $z_k = \xi \exp(kJt)$, where $\xi \in \mathbb{R}^{2N}$, $\frac{1}{2}|\xi|^2 = \rho_k$, $\phi'(\rho_k) = k$. As for the corresponding critical manifold Z_k, say, one has $Z_k = \{-J\dot{z}_k\}$ and hence $Z_k \simeq S^{2N}$. The points where f attains the minimum correspond to solutions of minimal period 2π, and hence to $k = 1$. To such a critical manifold Theorem 3.4 and Remark 3.5 can be applied, because $H_q(K_c) \simeq H_q(S^{2N})$.

In order to apply Theorem 2.1, we need to assume a further condition: $\phi''(\rho_k) \neq 0$. In fact the linearized equation at z_k is

$$-J\dot{v} = kv + \phi''(\rho_k)\langle v, z_k\rangle z_k. \tag{3.9}$$

Multiplying by z_k and integrating we get

$$\phi''(\rho_k) \int_0^T \langle v, z_k\rangle |z_k|^2 \, dt = \phi''(\rho_k)\, 2\rho_k \int_0^T \langle v, z_k\rangle \, dt = 0.$$

Now, if $\phi''(\rho_k) \neq 0$, it follows that $\operatorname{Ker} f''(-J\dot{z}_k)$ is a $(2N-1)$ dimensional space which coincides with the tangent space to Z_k.

Remark that if $\phi''(\rho_k) = 0$, then (2.2) is not satisfied, but Theorem 3.4 still applies even if $\phi''(\rho_1) = 0$.

We end this section with a short discussion on the applicability of Remark 2.7. This is the case, for example, if $H(z)$ is even in z and the perturbed system is like

$$-J\dot{z} = H'(z) + \varepsilon h(t, z), \tag{3.10}$$

with h T-periodic in t and even in z. In this case f_ε breaks the S^1 symmetry of f, but is still even. It follows that, in the setting of Theorem 3.6, for example, (3.10) will have at least *two pairs* of T-periodic solutions near $z_0(t+\theta)$, for ε small enough.

For other results on the persistence of solutions as consequence of the retention of some smaller symmetry, see, for example, [3, 6, 18].

4. Necessary Condition for Bifurcation

The problem discussed in Section 3 is actually a bifurcation one.

Precisely, we can take in the equation $f'(\varepsilon, u) = f'(\varepsilon, Pu + Qu) = 0$, Pu as parameter, so that the "trivial solution" is $\varepsilon = 0$, $Qu = 0$.

In this setting the question arises to find necessary conditions for Pu in Z to be a bifurcation point. The conditions we obtain are the same as those found in [1].

We shall consider f_ε as a functional $f(\varepsilon, u)$ from $\mathbb{R} \times E \to \mathbb{R}$($E$ as in Theorem (3.6)). We will denote the second partial derivative of $f(\varepsilon, u)$ by $D_{\varepsilon u} f$, $D_{uu} f$, etc. It results:

$$D_{\varepsilon u} f(\varepsilon, u)[u_1] = -\int_0^T \langle G''(u - \varepsilon h)h, u_1\rangle \, dt;$$

$$D_{uu} f(\varepsilon, u)[u_1, u_2] = \int_0^T \langle G''(u - \varepsilon h)\, u_1, u_2\rangle \, dt - \int_0^T \langle u_1, \mathscr{L} u_2\rangle \, dt.$$

Taking $w^* \in Z$, we set

$$A = (D_{\varepsilon u} f(0, w^*), D_{uu} f(0, w^*)) \in L(\mathbb{R} \times E, E).$$

Since $\operatorname{Ker} f''(w^*) = \mathbb{R}\dot{w}^*$, then it follows that the range of A is spanned by

$$D_{uu} f(0, w^*) E = \left\{ v \in E : \int_0^T \langle v, \dot{w}^* \rangle \, dt = 0 \right\}$$

and

$$D_{\varepsilon u} f(0, w^*) = -G''(w^*) h + \frac{1}{T} \int_0^T G''(w^*) \, d \, dt.$$

Set

$$V = \left\{ \alpha \dot{w}^*, \alpha \in \mathbb{R} : \alpha \int_0^T \langle G''(w^*) \, h, w^* \rangle \, dt = 0 \right\}$$

and denote by $z^* = G'(w^*) = \mathscr{L} w^* + \xi$ the solution of (H_0) corresponding to w^*. One has

$$\dot{w}^* = \frac{d}{dt} H'(z^*) = H''(z^*) \, \dot{z}^* = H''(z^*) \, J w^*,$$

hence

$$\int_0^T \langle G''(w^*) h, \dot{w}^* \rangle \, dt = \int_0^T \langle G''(w^*) h, H''(z^*) \, J w^* \rangle \, dt$$

$$= \int_0^T \langle h, J w^* \rangle \, dt = \int_0^T \langle h, \dot{z}^* \rangle \, dt.$$

Therefore $V = \mathbb{R}\dot{w}^*$ if $\int_0^T \langle h, \dot{z}^* \rangle \, dt = 0$, otherwise $V = \{0\}$. As in [8] one can show:

LEMMA 4.1. *There is a neighborhood $\mathscr{U}$ in $\mathbb{R} \times E$ of $(0, w^*)$ such that, setting*

$$S = \{ (\varepsilon, w) \in \mathbb{R} \times E : f'(\varepsilon, w) \in V \},$$

$S \cap \mathscr{U}$ is a C^∞-submanifold in $\mathbb{R} \times E$ of dimension $1 + \dim V$.

As a consequence one has:

THEOREM 4.2. *Let $\int_0^T \langle h, \dot{z}^* \rangle \, dt \neq 0$. Then there is a neighborhood $\mathscr{U}$ of $(0, w^*)$ in $\mathbb{R} \times E$ such that $(\varepsilon, u) \in \mathscr{U}$ and $f'(\varepsilon, u) = 0$ implies $\varepsilon = 0$ and $u \in Z$.*

Proof. If $f'(\varepsilon, u)=0$, then $(\varepsilon, u)\in S$. In the present case $V=\{0\}$ and hence $S\cap\mathscr{U}$ is one-dimensional in a suitable neighborhood $\mathscr{U}$ according to Lemma 4.1. Since Z is 1-dimensional and $\tilde{Z}:=\{(0,u): u\in Z\}$ is contained in S, then $S\cap\mathscr{U}=\tilde{Z}\cap\mathscr{U}$, taking $\mathscr{U}$ possibly smaller. This proves the result. In other words, if $\int_0^T \langle h, z^*\rangle\, dt\neq 0$, there are no solutions of (H_ε) near z^* for $\varepsilon\in 0$. ∎

5. Remarks on Stability

We now return to the results of Section 3, with the purpose of discussing the stability of the periodic solutions of the perturbed system (H_ε). We shall confine ourselves to the simple situation of Theorem 3.6, where, for $\varepsilon=0$, we dealt with a nondegenerate T-periodic solution of (H_0). The argument above extends readily to the more complicated situations of Theorem 3.7 and Example 3.8.

Theorem 5.1. *Assume* (A5) *and let z_0 be a nondegenerate T-periodic solution of (H_0). Let $(1, 1, \lambda_3,..., \lambda_{2N})$ be the Floquet multipliers of z_0, with $\lambda_i\neq 1$ for $i\geqslant 3$.*

Assume that, for $0<|\varepsilon|<\bar{\varepsilon}$, the T-periodic solutions of the forced system (H_ε) near z_0 are nondegenerate. Then one can find two of them, z_ε^1 and z_ε^2, with Floquet multipliers $(\lambda_1^j(\varepsilon),..., \lambda_{2N}^j(\varepsilon))$, $j=1, 2$, such that

$$|\lambda_i^j(\varepsilon)-\lambda_i|\to 0 \qquad \textit{when } \varepsilon\to 0,\ \forall(i,j); \tag{5.1}$$

$$\lambda_1^1(\varepsilon)\notin\mathbb{R},\ |\lambda_1^1(\varepsilon)|=1 \qquad \textit{and} \qquad \lambda_2^1(\varepsilon)=\bar{\lambda}_1^1(\varepsilon) \qquad \forall\varepsilon\neq 0;^1 \tag{5.2}$$

$$\lambda_1^2(\varepsilon)\in\mathbb{R},\ \lambda_1^2(\varepsilon)>1 \qquad \textit{and} \qquad \lambda_2^2(\varepsilon)=\lambda_1^2(\varepsilon)^{-1} \qquad \forall\varepsilon\neq 0. \tag{5.3}$$

Let us clear up the result before proceeding to the proof.

First, note that the work "nondegenerate" does not have the same meaning when it is applied to z_0 or to z_ε^1 and z_ε^2. In the autonomous, S^1 invariant situation, it means that the linearized equation around z_0, say $\dot{y}=JH''(z_0(t))\,y$, has no other T-periodic solution than $y=\dot{z}_0$ (see Section 3). In the perturbed, nonautonomous case, it means that the linearized equation has no T-periodic solutions, so $\lambda_i^j(\varepsilon)\neq 1$ for all (i,j).

Theorem 3.6 tells us that the forced system (H_ε) has several T-periodic solutions z_ε^j, $1\leqslant j\leqslant k$, with $z_\varepsilon^j\to z_0$ when $\varepsilon\to 0$. We associate with each of them the resolvent R_ε^j of the linearized system, that is, the solution of $\dot{R}_\varepsilon^j=J_{H_\varepsilon''}(z_\varepsilon^j(t))\,R_\varepsilon^j$, with $R_\varepsilon^j(0)=I$. The Floquet multipliers $\lambda^j(\varepsilon)$ are just the eigenvalues of $R_\varepsilon^j(T)$, and property (5.1) follows immediately from known continuity results.

[1] $\bar{\beta}$ denotes the complex conjugate of β.

We can say something more if, for instance, all the λ_i, $i \geqslant 3$, are simple eigenvalues, depending continuously on ε. Since the matrices $R_\varepsilon^j(T)$ are symplectic, it is known that a simple eigenvalue cannot leave the unit circle: if $|\lambda_i| = 1$. $i \geqslant 3$, then $|\lambda_i^j(\varepsilon)| = 1$.

On the other hand, if $|\lambda_i| \neq 1$, then $|\lambda_i^j(\varepsilon)| \neq 1$ by continuity.

This means that the only point where the eigenvalues can leave the unit circle is $\lambda_1 = 1 = \lambda_2$. We know that $R_\varepsilon^j(T)$ is symplectic, so if μ is an eigenvalue so are $\bar{\mu}$ and μ^{-1}. It follows that the two only possible situations, when $\lambda_1^j(\varepsilon)$ and $\lambda_2^j(\varepsilon)$ are close to 1, are given by (5.2) and (5.3). Theorem 5.1 states that both have to occur. So there must be a perturbed solutions where λ_1 and λ_2 remain on the unit circle, and another one where they leave it.

If, for instance, the solution z_0 was orbitally linearly stable ($|\lambda_i| = 1$ for all i), and all the eigenvalues λ_i for $i \geqslant 3$ were simple, then z_ε^1 remains linearly stable, while z_ε^2 becames unstable. This is in accordance with the general results from bifurcation theory.

The proof of Theorem 5.1 relies on the following lemma and on the index theory developed in [9]. In that paper, it is shown how a finite dimensional C^2 model can be built for the functionals f and f_ε which have been defined in Section 4. Taking this, into account, we will freely use Morse theory as if the underlying space E were finite-dimensional.

LEMMA 5.2. *There must be two values j_1 and j_2 such that the Morse indices of $z_\varepsilon^{j_1}$ and $z_\varepsilon^{j_2}$ differ by* 1.

Proof. Set $u_0 = -J\dot{z}_0$ and $Z = \{A_\theta u_0 : \theta \in S^1\}$. Among the critical points of f_ε near Z, we take u_ε^1 and u_ε^2 with

$$f_\varepsilon(u_\varepsilon^1) = \max\{f_\varepsilon(w) : w \in Z_\varepsilon\},$$

$$f_\varepsilon(u_\varepsilon^2) = \min\{f_\varepsilon(w) : w \in Z_\varepsilon\}.$$

Let $v_i^* \in Z$ be such that $v_i^* = P(u_\varepsilon^i)$; see notation in Section 2. Recall that (see [2]) Z is nondegenerate if $\forall z \in Z\ f''(z)$ is nondegenerate on $(T_z A)^\perp$. The Morse index of Z (equal to the Morse index of z_0) is the maximal dimension of $\{v \in (T_z Z)^\perp : f''(z)[v, v] < 0\}$. Let l be the Morse index of Z. We set

$$\Gamma_\varepsilon^i = T_{u_\varepsilon^i} Z_\varepsilon \qquad \text{and} \qquad E = \Gamma_\varepsilon^i \oplus (\Gamma_\varepsilon^i)^\perp.$$

By continuity one has

$$\dim\{w \in (\Gamma_\varepsilon^i)^\perp : f_\varepsilon''(u_\varepsilon^i)[w, w] < 0\} = l.$$

Take now u_ε^1. Since u_ε^1 is the nondegenerate point of maximum of f_ε on Z_ε, it follows that $f_\varepsilon''(u_\varepsilon^1)$ is negative definite on Γ_ε^1 (which is 1-dimensional). Hence $\operatorname{ind}(u_\varepsilon^1) = l+1$.

As for u_ε^2, $f_\varepsilon''(u_\varepsilon^2)$ is positive defined on Γ_ε^2 and then one has $\operatorname{ind}(u_\varepsilon^2) = l$. ∎

Remark 5.3. Lemma 5.2 would follow directly from the arguments sketched in Section 1, after Remark 1.3, if we assume f and f_ε satisfy inequalities (1.8) (with $g = f_\varepsilon$). This is the case if H satisfies (A1)–(A4); see Lemma 3.2 and the proof of Theorem 3.4.

We now use the index theory of [9], slightly adapted to nonautonomous systems. With every $\omega \in \mathbb{C}$ such that $|\omega| = 1$, we associate the complex space $H_\omega^1 = \{ y \in H^1([0, T]; \mathbb{C}^{2N}) \colon y(T) = \omega y(0) \}$ and the hermitian form

$$(Q_\varepsilon^j y, z) = \int_0^T \{ -\langle y, \mathscr{L} z \rangle + \langle G''(u_\varepsilon^j(t) - \varepsilon h(t))\, y, z \rangle \}\, dt.$$

We define $m_\varepsilon^j(\omega)$ to be the index of $(Q_\varepsilon^j y, z)$ on H_ω^1. In particular, $m_\varepsilon^j(1)$ is the index of $(f_\varepsilon''(u_\varepsilon^j)\, y, z)$ on E. It is just the Morse index of the critical points u_ε^j of f_ε on E.

Similarly, $m_\varepsilon^j(-1)$ is the index of the quadratic form $(Q_\varepsilon^j y, z)$ on the space of antiperiodic functions

$$H_{-1}^1 = \{ y \in H^1([0, T]; \mathbb{C}^{2N}) \colon y(0) + y(T) = 0 \}.$$

More generally, m_ε^j is an integer valued function on the unit circle, which can be constructed as follows (see [9]):

(a) the points of discontinuity are the $\lambda_i(\varepsilon)$ with $|\lambda_i(\varepsilon)| = 1$;

(b) if $\lambda_i(\varepsilon)$, with $|\lambda_i(\varepsilon)| = 1$, is a Floquet multiplier of Krein type (p, q), then the jump of m_ε^j across $\lambda_i(\varepsilon)$ is equal to $(q-p)$:

$$\lim_{\theta \to 0^+} \{ m_\varepsilon^j(e^{i\theta} \lambda_i(\varepsilon)) - m_\varepsilon^j(e^{-i\theta} \lambda_i(\varepsilon)) \} = q - p.$$

Now choose ω on the unit circle, with $\omega \neq \lambda_i$ for all i. By continuity, $\omega \neq \lambda_i(\varepsilon)$ for $|\varepsilon| < \bar{\varepsilon}$. It follows that $(Q_\varepsilon^j y, z)$ will be nondegenerate on H_ω^1 for all $|\varepsilon| < \bar{\varepsilon}$. But then the index cannot change, which proves that $m_\varepsilon^j(\omega) = m_0(\omega)$ for $|\varepsilon| < \bar{\varepsilon}$.

So we have found a point ω where the value $m_\varepsilon^j(\omega)$ does not depend on ε. Starting from this point, we may figure out $m_\varepsilon^j(1)$, using the rules (a) and (b). We first show that the following two cases are impossible.

Case 1. The $\lambda_1^j(\varepsilon)$, $1 \leqslant j \leqslant k$, all leave the unit circle. Then so do the $\lambda_2^j(\varepsilon) = (\lambda_1^j(\varepsilon))^{-1}$. The only jumps between $m_\varepsilon^j(\omega)$ and $m_\varepsilon^j(1)$ will come from the values $\lambda_i^j(\varepsilon)$ which lie on the unit circle, between ω and 1. But the λ_i then also lie on the unit circle, between ω and 1, and the jumps, by rule (b), are the same for $\lambda_i^j(\varepsilon)$ and λ_i. Indeed, if the Krein type is $(1, 0)$ or $(0, 1)$, the Floquet multiplier remains on the unit circle and its Krein type does not change. If the Krein type is (p, q), the Floquet multiplier may split into $(p+q)$ simple multipliers, some of which may leave the unit circle, but in such a way that the balance $q-p$ is not affected. Since $m_\varepsilon^j(\omega) = m_0(\omega)$ for all $|\varepsilon| < \bar{\varepsilon}$, and since the jumps between ω and 1 are the same, rule (b) tells us that the $m_\varepsilon^j(1)$ are the same for all j and all $\varepsilon \neq 0$ (the rule for $m_0(1)$ is different; see [9]). This contradicts Lemma 5.2.

Case 2. The $\lambda_1^j(\varepsilon)$, $1 \leqslant j \leqslant k$, all stay on the unit circle. Then so do the $\lambda_2^j(\varepsilon) = \bar{\lambda}_1^j(\varepsilon)$. The jumps between $m_\varepsilon^j(\omega)$ and $m_\varepsilon^j(1)$ will come from the $\lambda_i^j(\varepsilon)$, $i \geqslant 3$, which lie on the unit circle between ω and 1. The contribution of the $\lambda_1^j(\varepsilon)$ or $\lambda_2^j(\varepsilon)$ is either $+1$ or -1, depending on its Krein type, which is either $(1, 0)$ or $(0, 1)$ since it must be a simple Floquet multiplier. Since $m_\varepsilon^j(\omega) = m_0(\omega)$, we see that all the $m_\varepsilon^j(1)$, $1 \leqslant j \leqslant k$, $0 < |\varepsilon| < \bar{\varepsilon}$, must be the same modulo 2. This contradicts Lemma 5.2 again.

Since cases 1 and 2 are impossible, at least one of the $\lambda_i^j(\varepsilon)$ leaves the unit circle, and at least another one stays on it. This concludes the proof of Theorem 5.1. ∎

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